Chapter 12 Differentiation

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1. The area of a sector of a circle of radius $r\ cm$ is $36\ cm^2$.
   a. Show that the perimeter, *P* cm, of the sector is such that $P = 2r + \frac{72}{r}$. [3]

$A = \frac{1}{2}r^2\theta$

$36 = \frac{1}{2} \times r^2 \times \theta$

$72 = r^2\theta$

$\theta = \frac{72}{r^2}$

$P = r\theta + 2r$

$= \cancel{r} \times \frac{72}{r^{\cancel{2}}} + 2r$

$= \frac{72}{r} + 2r$ (shown)

   b. Hence, given that *r* can vary, find the stationary value of *P* and determine its nature. [4]

$\frac{dP}{dr} = -\frac{72}{r^2} + 2$

$-\frac{72}{r^2} + 2 = 0$

$-\frac{72}{r^2} = -2$

$\frac{72}{r^2} = 2$

$r^2 = 36$

$r = 6$ cm

$P = \frac{72}{r} + 2r$

$= 24$ cm

$\frac{d^2P}{dr^2} = \frac{144}{r^3}$

$= \frac{2}{3} > 0$ ∴ minimum value

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2. (a) Given that $y = x\sqrt{x^2 + 1}$, show that $\frac{dy}{dx} = \frac{ax^2+b}{(x^2+1)^p}$, where a, b and p are positive constants.

[4]

$$\frac{dy}{dx} = (x^2+1)^{1/2} + \frac{1}{\cancel{2}}(x^2+1)^{-1/2} \times \cancel{2}x \times x$$

$$= (x^2+1)^{1/2} + \frac{x^2}{(x^2+1)^{1/2}}$$

$$= \frac{(x^2+1) + x^2}{(x^2+1)^{1/2}} = \frac{2x^2+1}{(x^2+1)^{1/2}}$$

(b) Explain why the graph of $y = x\sqrt{x^2 + 1}$ has no stationary points.

[2]

Assume y has a stationary point.

$$\frac{dy}{dx} = 0$$

$$\frac{2x^2+1}{(x^2+1)^{1/2}} = 0$$

$$2x^2+1 = 0$$

$$2x^2 = -1$$

$$x^2 = -\frac{1}{2}$$

(reject)

$\therefore \frac{dy}{dx} \neq 0$

$\therefore$ no stationary point.

3. It is given that $y = (x^2 + 1)(2x - 3)^{\frac{1}{2}}$.

(i) Show that $\frac{dy}{dx} = \frac{Px^2+Qx+1}{(2x-3)^{\frac{1}{2}}}$, where P and Q are integers. [5]

$$\frac{dy}{dx} = 2x(2x-3)^{1/2} + \frac{1}{\cancel{2}} \times \cancel{2}(2x-3)^{-1/2}(x^2+1)$$

$$= 2x(2x-3)^{1/2} + \frac{(x^2+1)}{(2x-3)^{1/2}}$$

$$= \frac{2x(2x-3) + x^2+1}{(2x-3)^{1/2}}$$

$$= \frac{4x^2-6x+x^2+1}{(2x-3)^{1/2}} = \frac{5x^2-6x+1}{(2x-3)^{1/2}}$$

(ii) Hence find the equation of the normal to the curve $y = (x^2 + 1)(2x - 3)^{\frac{1}{2}}$ at the point where x = 2, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[4]

$$\frac{dy}{dx} = \frac{5x^2 - 6x + 1}{(2x-3)^{1/2}} = 20 - 12 + 1$$

$$= 9$$

$$m_{normal} = -\frac{1}{9}$$

$$y = (x^2+1)(2x-3)^{1/2}$$

$$x = 2, \; y = 5$$

$$y = -\frac{1}{9}x + C$$

$$5 = -\frac{2}{9} + C$$

$$C = \frac{45 + 2}{9}$$

$$C = \frac{47}{9}$$

$$y = -\frac{1}{9}x + \frac{47}{9}$$

$$9y = -x + 47$$

$$x + 9y - 47 = 0$$

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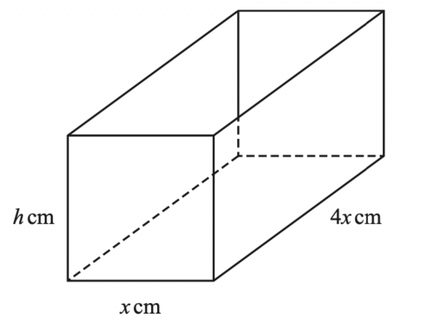
4.



The diagram shows an open container in the shape of a cuboid of width *x* cm, length 4*x* cm and height *h* cm. The volume of the container is $800cm^3$.

a. Show that the external surface area, $S\ cm^2$, of the open container is such that $S = 4x^2 + \frac{2000}{x}$.

[4]

$$S.A = 2hx + 8hx + 4x^2$$
$$= 10hx + 4x^2$$

$$V = h \times 4x \times x$$
$$800 = 4x^2 h$$
$$h = \frac{200}{x^2}$$

$$SA = 10 \times \frac{200}{x^{\cancel{2}}} \times \cancel{x} + 4x^2$$

$$= \frac{2000}{x} + 4x^2$$

(shown)

b. Given that x can vary, find the stationary value of S and determine its nature. [5]

$$S = \frac{2000}{x} + 4x^2$$

$$\frac{dS}{dx} = -\frac{2000}{x^2} + 8x$$

$$\frac{dS}{dx} = 0$$

$$-\frac{2000}{x^2} + 8x = 0$$

$$\frac{-2000}{x^2} = -8x$$

$$2000 = 8x^3$$

$$250 = x^3$$

$$x = \sqrt[3]{250}$$

$$x = 6.3 \text{ cm}$$

$$S = \frac{2000}{x} + 4x^2$$

$$= \frac{2000}{6.3} + 4(6.3)^2$$

$$= 476.22$$

$$\frac{d^2S}{dx^2} = \frac{4000}{x^3} + 8$$

$$= \frac{4000}{(6.3)^3} + 8$$

$$= 23.99 > 0 \quad \text{minimum point}$$

5. The normal to the curve $y = (x - 2)(3x + 1)^{\frac{2}{3}}$ at the point where $x = \frac{7}{3}$, meets the y-axis at the point P. Find the exact coordinates of the point P.

[7]

$$y = (x-2)(3x+1)^{2/3}$$

$$\frac{dy}{dx} = (3x+1)^{2/3} + \frac{2}{\cancel{3}} \times \cancel{3}\,(3x+1)^{-1/3}(x-2)$$

$$= (3x+1)^{2/3} + \frac{2x-4}{(3x+1)^{1/3}}$$

$$x = \frac{7}{3}, \quad \frac{dy}{dx} = 4 + \frac{\frac{14}{3} - 4}{2} = \frac{13}{3}$$

$$m_{normal} = -\frac{3}{13}$$

$$x = \frac{7}{3}, \quad y = (x-2)(3x+1)^{2/3}$$

$$= \frac{4}{3}$$

$$y = -\frac{3}{13}x + c$$

$$\frac{4}{3} = -\frac{7}{13} + c$$

$$c = \frac{4}{3} + \frac{7}{13}$$

$$c = \frac{73}{39}$$

$$y = -\frac{3}{13}x + \frac{73}{39}$$

$$\therefore P\left(0, \frac{73}{39}\right)$$

6. .A circle has diameter *x* which is increasing at a constant rate of 0.01 $cm\,s^{-1}$. Find the exact rate of change of the area of the circle when *x* = 6 cm.

$A = \pi \frac{x^2}{4}$ $\quad \frac{dx}{dt} = 0.01 \quad \frac{dA}{dt} = ?$ [5]

$\frac{dA}{dx} = \frac{\pi x}{2} = 3\pi$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= 3\pi \times 0.01$

$= 0.03\pi$

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7. A curve has equation $y = (3x - 5)^2 - 2x$.

a. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [4]

$\frac{dy}{dx} = 2(3x-5) \times 3 - 2$

$= 6(3x-5) - 2$

$= 18x - 30 - 2$

$= 18x - 32$

$\frac{d^2y}{dx^2} = 18$

b. Find the exact value of the *x*-coordinate of each of the stationary points of the curve. [2]

$\frac{dy}{dx} = 0$

$18x - 32 = 0$

$18x = 32$

$x = \frac{16}{9}$

c. Use the second derivative test to determine the nature of each of the stationary points. [2]

$\frac{d^2y}{dx^2} = 18 > 0 \quad \therefore$ minimum value

8. Find the equation of the normal to the curve $y = \sqrt{8x + 5}$ at the point where $x = \frac{1}{2}$ giving your answer in the form $ax + by + c = 0$, where *a*, *b* and *c* are integers.

[5]

$y = (8x+5)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{\cancel{2}_{1}}(8x+5)^{-\frac{1}{2}} \times \cancel{8}\,4$

$= 4(8x+5)^{-\frac{1}{2}}$

$= 4(8 \times \frac{1}{2} + 5)^{-\frac{1}{2}}$

$= \frac{4}{\sqrt{9}} = \frac{4}{3}$

$m_{normal} = -\frac{3}{4}$

$y = (8x+5)^{\frac{1}{2}}$

$y = 3$

$y = -\frac{3}{4}x + c$

$3 = -\frac{3}{8} + c$

$c = \frac{24+3}{8} = \frac{27}{8}$

$y = -\frac{3}{4}x + \frac{27}{8}$

$8y = -6x + 27$

$8y + 6x - 27 = 0$

9. A solid circular cylinder has a base radius of *r* cm and a height of *h* cm. The cylinder has a volume of $1200\pi\ cm^3$ and a total surface area of $S\ cm^2$.

a. Show that $S = 2\pi r^2 + \frac{2400\pi}{r}$. [3]

$S = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + \frac{2\pi \times 1200}{r}$

$= 2\pi r^2 + \frac{2400\pi}{r}$

$V = \pi r^2 h$

$1200\pi = \pi r^2 h$

$1200 = r^2 h$

$h = \frac{1200}{r^2}$

b. Given that h and r can vary, find the stationary value of S and determine its nature. [5]

$$S = 2\pi r^2 + \frac{2400\pi}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{2400\pi}{r^2}$$

$$\frac{dS}{dr} = 0$$

$$4\cancel{\pi} r = \frac{2400\cancel{\pi}}{r^2}$$

$$r^3 = 600$$

$$r = \sqrt[3]{600}$$

$$= 8.43$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4800\pi}{r^3}$$

$$= 37.74 > 0 \quad \therefore \text{ minimum value}$$

$$S = 2\pi r^2 + \frac{2400\pi}{r}$$

$$= 2\pi(8.43)^2 + \frac{2400\pi}{8.43}$$

$$= 1340.92$$

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10. (i) Differentiate $y = (3x^2 - 1)^{\frac{-1}{3}}$ with respect to x. [2]

$$\frac{dy}{dx} = -\frac{1}{\cancel{3}}(3x^2-1)^{-4/3} \times \cancel{6}^{\,2}x$$

$$= -2x(3x^2-1)^{-4/3}$$

(ii) Find the approximate change in y as x increases from $\underset{x}{\sqrt{3}}$ to $\sqrt{3} + \underset{\delta x}{p}$, where p is small. [1]

$$\frac{dy}{dx} = -2\sqrt{3}\,(9-1)^{-4/3}$$
$$= -2\sqrt{3}\,(8)^{-4/3}$$
$$= -2\sqrt{3}\,(2^{-4})$$

$$= -\frac{\sqrt{3}}{8}$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$\delta y \approx -\frac{\sqrt{3}}{8}p$$

(iii) Find the equation of the normal to the curve $y = (3x^2 - 1)^{\frac{-1}{3}}$ at the point where $x = \sqrt{3}$. [3]

$$m_{normal} = \frac{8}{\sqrt{3}}$$

$$y = (9-1)^{-1/3}$$
$$= (8)^{-1/3} = (2^3)^{-1/3}$$
$$= 2^{-1} = \frac{1}{2}$$

$$y = \frac{8}{\sqrt{3}}x + c$$

$$\frac{1}{2} = \frac{8}{\cancel{\sqrt{3}}} \times \cancel{\sqrt{3}} + c$$

$$c = -\frac{15}{2}$$

$$y = \frac{8}{\sqrt{3}}x - \frac{15}{2}$$

$= k(x+1)^{-2}$

11. At the point where $x = 1$ on the curve $y = \frac{k}{(x+1)^2}$, the normal has a gradient of $\frac{1}{3}$.

   a. Find the value of the constant k.

$\frac{dy}{dx} = -2k(x+1)^{-3}$

[4]

$= \frac{-2k}{(x+1)^3} = \frac{-2k}{8} = -\frac{k}{4}$

$m_{normal} = \frac{4}{k}$

$\frac{4}{k} = \frac{1}{3}$

$k = 12$

   b. Using your value of k, find the equation of the tangent to the curve at $x = 2$.

$\frac{dy}{dx} = -2k(x+1)^{-3}$

$y = \frac{k}{(x+1)^2}$

[3]

$= -24(3)^{-3}$

$= \frac{12}{9} = \frac{4}{3}$

$= \frac{-24}{27} = -\frac{8}{9}$

$y = -\frac{8}{9}x + c$

$\frac{4}{3} = -\frac{16}{9} + c$

$c = \frac{12+16}{9}$

$\therefore y = -\frac{8}{9}x + \frac{28}{9}$

$c = \frac{28}{9}$